

STABILITY OF PLANE MAGNETOHYDRODYNAMIC  
COUETTE FLOW WITH ASYMMETRICAL VELOCITY PROFILE

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The hydrodynamic stability of plane magnetohydrodynamic Couette flow with asymmetrical velocity profile formed by a transverse magnetic field is investigated within the framework of the linear theory. The complete spectrum of the small perturbations is studied for the characteristic Hartmann numbers. The perturbations are classified in accordance with their phase velocity at large wave numbers. It is established that the stability of the flow is controlled by only one type of perturbations. The critical parameters of the problem are determined. The instability in question recalls the instability of Hartmann flow against asymmetrical perturbations.

1. We consider plane-parallel flow of a viscous incompressible conducting liquid in a transverse magnetic field, produced by the motion of the upper plate. The expression for the velocity profile is given in [1]

$$u = \frac{\text{sh } Gy}{\text{sh } G} \quad (1.1)$$

Here  $u$  is the velocity profile, and  $G$  is the Hartmann number. The unit length was chosen to be the width of the channel, and the unit velocity that of the upper plate  $y = 1$ . The boundary conditions used in the derivation of (1.1) are

$$H_x = 0, \quad dH_x/dy = 0, \quad u = 0 \quad \text{for } y = 0; \quad u = 1 \quad \text{for } y = 1$$

( $H_x$  is the longitudinal component of the magnetic field).

This flow is one of the simplest laminar motions of a liquid in magnetohydrodynamics. It can serve as a certain approximation to the boundary-layer problem [1], and the corresponding velocity profile can serve as a model for the determination of certain general characteristics of the hydrodynamic stability of flows with symmetrical velocity profiles.

We assume the magnetic Reynolds numbers to be small. This requirement is satisfied by most flows realized in experiments. Then the problem of hydrodynamic stability reduces to an analysis of the eigenvalue spectrum of the modified Orr-Sommerfeld equation [2]

$$\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi = i\alpha R[(u - c)(\varphi'' - \alpha^2\varphi) - u''\varphi] + G^2\varphi'' \quad (1.2)$$

with homogeneous boundary conditions

$$\varphi(y) = \varphi'(y) = 0 \quad \text{for } y = 0, 1 \quad (1.3)$$

Here  $\varphi(y)$  is the amplitude of the stream function of the perturbation;  $\alpha$  is the wave number;  $R$  is the Reynolds number;  $c = X + iY$  is the complex phase velocity of the perturbation and the eigenvalue of the problem. Positive and negative  $Y$  correspond to perturbations that increase and attenuate with time, respectively. The corresponding decrement is  $\alpha Y$ .

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Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 12-18, May-June, 1971. Original article submitted May 7, 1970.

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Another form of plane magnetohydrodynamic Couette flow was investigated in [3, 4]. The problem studied below differs from that considered earlier in that additional boundary conditions are posed on the channel axis of the flow with asymmetrical profile, namely the adhesion conditions for the perturbations.

Much attention will be paid below to a study of the complete spectrum of the small perturbations, which is of interest in the analysis of the time behavior of an arbitrary perturbation. Information concerning the complete spectrum may be necessary also for the development of a nonlinear theory.

2. When  $G$ ,  $R$ , and  $\alpha$  are fixed, the solution of the eigenvalue problem (1.2) and (1.3) defines a denumerable set of quantities  $c_n$ . At small  $\alpha R$ , the eigenvalues are obtained by perturbation theory in analogy with the case investigated in [5]. The solution of the problem is sought in the form of a series in powers of  $\alpha R$ . For our purposes it suffices to confine ourselves to the first two approximations, which determine  $Y_n$  and  $X_n$ , respectively. Omitting the cumbersome intermediate transformations, we present the required expressions. The formula for  $Y_n$  is

$$Y_n = -4\lambda_n / \alpha R \quad (2.1)$$

where  $\lambda_n$  is determined from the equations

$$\delta \operatorname{tg} \delta = -k \operatorname{th} k, \quad \delta \operatorname{ctg} \delta = k \operatorname{cth} k \quad (2.2)$$

In (2.2) we have introduced the notation

$$\begin{aligned} \delta &= 1/2 \left[ \sqrt{1/4 (4\lambda_n - G^2)^2 + \alpha^2 G^2} - \alpha^2 - 1/2 (G^2 - 4\lambda_n) \right]^{1/2} \\ k &= 1/2 \left[ \sqrt{1/4 (4\lambda_n - G^2)^2 + \alpha^2 G^2} + \alpha^2 + 1/2 (G^2 - 4\lambda_n) \right]^{1/2} \end{aligned}$$

If  $\alpha$  is small, then it is easy to find from (2.2)

$$\lambda_n = \kappa_n + 1/4 G^2$$

Here  $\kappa_n = 1/4 \pi^2 (n+1)^2$  for  $n = 1, 3, 5, \dots$ , and  $\kappa_n$  is a solution of the equation  $\tan \sqrt{\kappa_n} = \sqrt{\kappa_n}$  for  $n = 2, 4, 6, \dots$  (The zeroth solution is excluded from consideration.) Thus, at sufficiently small  $\alpha$  and small  $\alpha R$  and we have the asymptotic expression

$$Y_n = -\frac{4\kappa_n + G^2}{\alpha R} \quad (2.3)$$

The spectral numbering at small  $\alpha$  is in increasing order of  $|Y_n|$ , and for arbitrary  $\alpha$  it follows the order of the eigenvalues at small  $\alpha$ .

Bearing in mind an approximation of zeroth order in  $\alpha$ , we obtain, furthermore,

$$\begin{aligned} X_n &= \langle u \rangle - \frac{GM}{4} \left[ \frac{2}{\lambda_n} \left( 1 + \frac{G^2}{2\kappa_n} \right) - \frac{3}{\kappa_n} - \frac{\lambda_n}{\kappa_n (\lambda_n + 3\kappa_n)} \right] \\ &\quad (n = 1, 3, 5, \dots) \\ X_n &= \langle u \rangle - \frac{G^3}{2\lambda_n^2} \left( 1 + \frac{G^2}{2\kappa_n} \right) [(2 + \lambda_n) M - G] - \\ &\quad - \frac{\lambda_n}{2\kappa_n (\lambda_n + 3\kappa_n)} \left( \frac{GM}{2\kappa_n} - \frac{GM}{2} + 4 \right) + \frac{M}{2\kappa_n} \left( \frac{3G}{2} + \frac{10}{G} \right) - \frac{2}{\kappa_n} + \frac{GM}{4\kappa_n^2} \\ &\quad (n = 2, 4, 6, \dots) \\ (M &= \operatorname{th} 1/2 G, \quad \langle u \rangle = M/G) \end{aligned} \quad (2.4)$$

Here  $\langle u \rangle$  is the stream velocity averaged over the channel cross section, which makes the main contribution in the expressions for  $X_n$ . As  $n \rightarrow \infty$  (with all other parameters fixed),  $X_n$  rapidly approaches the value of  $\langle u \rangle$ .

At large  $\alpha$  we have  $|c_n| \gg 1$ , and, therefore, to find  $Y_n$  in the first approximation we can neglect the shape of the profile, just as in the case of small  $\alpha$ , and obtain

$$Y_n = -\alpha / R \quad (n \ll \alpha) \quad (2.5)$$

The magnetic field does not play any role here, in accordance with the general concepts. (The values of  $G$  are assumed to be bounded.)

It is difficult to obtain an analytic estimate for  $X_n$  at large  $\alpha$ .

The eigenvalues corresponding to large  $n$  are close, at finite Reynolds numbers, to the corresponding values in a liquid at rest [see Eq. (2.1)].

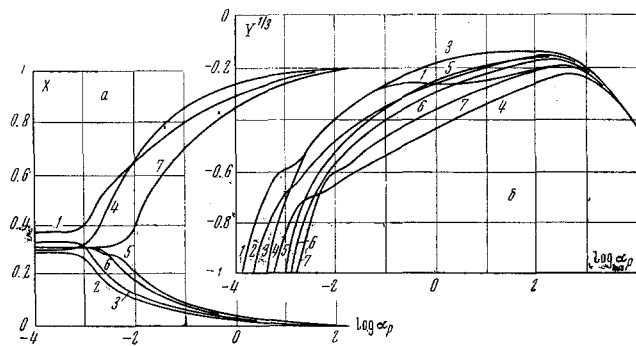


Fig. 1

An investigation of the spectrum in the interval between the asymptotic expressions (2.3), (2.4), and (2.5) was carried out with the aid of a numerical method developed in [6-8]. The numerical calculations were performed with a BESM-6 computer. The eigenvalues were obtained with specified accuracy (three significant figures). The control calculations were performed with the Poiseuille flow as an example, and gave good agreement with the numerical results obtained in [9].

3. Let us consider the behavior of small perturbations as a function of  $\alpha$  and  $n$  for the characteristic Hartmann numbers at fixed  $R = 2 \cdot 10^5$ . The limiting case  $G = 0$  was investigated by many workers. The latest interesting results were obtained in [10, 11], where the stability of plane Couette flow is apparently covered exhaustively.

With increasing  $G$ , an appreciable realignment of the spectrum takes place. In particular, the points of multiplicity of the eigenvalues, which are present only in the limiting case  $G = 0$ , vanish.

Figure 1 shows plots of  $X_n(\alpha)$  and  $Y_n(\alpha)$  at  $G = 3$  for the first seven spectral numbers in the entire range between the asymptotic expressions (here and below,  $\alpha_p = \alpha/2$ ). When  $\alpha$  increases, the phase velocity of the perturbations tends either to the velocity of the lower wall or to the velocity of the upper wall. Since this takes place at all Hartmann numbers, we shall classify the perturbations as upper and lower wall perturbations, in accordance with the behavior of their phase velocity at large wave numbers. In this case the spectral numbers  $n = 1, 4, 7, \dots$  correspond to the upper wall perturbations, and the spectral numbers  $n = 2, 3, 5, 6, \dots$  correspond to the lower wall perturbations.

There is a typical difference between the behavior of the upper and lower wall functions  $Y_n(\alpha)$  beyond the asymptotic region at small  $\alpha$ . The hydrodynamics of the stream comes into play here primarily in the upper wall perturbations, whose damping decrement increases in comparison with the case of a liquid at rest. The net result is a realignment of the spectrum immediately beyond the region of applicability of formula (2.3). Intersections in the spectrum occur also at larger  $\alpha$ . The minimum damping decrement is not determined everywhere by the first eigenvalue, in contrast to the Couette flow. At the given problem parameters, the flow is stable. In the region of "dangerous" wave numbers (i.e., values of the order of the reciprocal characteristic flow dimension), the function  $Y_1(\alpha)$  has a weak local maximum.

To gain an idea of the spectrum of the small perturbations at "medium" Hartmann numbers, numerical calculations were performed at  $G = 6$ . In this case the perturbation with  $n = 1$  is the upper wall perturbation, and those with  $n = 2, 3, 4, 5$  are the lower wall perturbations. We call attention to the asymmetry in the subdivision of the perturbations into upper and lower ones.

The singularity noted above in the influence of the hydrodynamics on the damping decrement at small  $\alpha$  holds true also in the present case. A characteristic behavior is exhibited by the lower wall functions  $Y_n(\alpha)$ , which do not intersect for all  $\alpha$ . The function  $Y_1(\alpha)$  has a pronounced local maximum at  $\alpha = 0.55$ , which, however, lies lower in the  $Y$  plane than the function  $Y_2(\alpha)$ . In the range  $\alpha = 1-10^3$ , the damping decrement determined by  $Y_1(\alpha)$  is much larger than that determined from the indicated lower wall perturbations.

With increasing  $n$ , the region of applicability of the asymptotic expressions (2.3) and (2.4) increases. The asymptotic behavior of the lower wall functions at large  $\alpha$  becomes obvious earlier than for the upper wall function. These general circumstances occur also in the other considered cases.

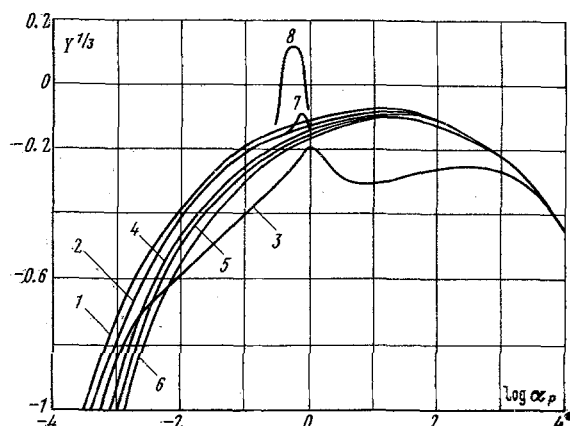


Fig. 2

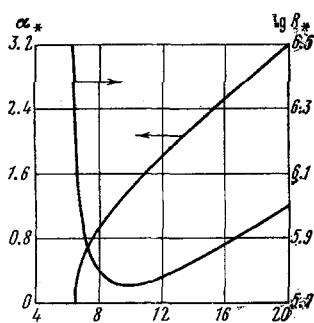


Fig. 3

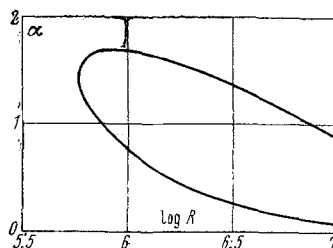


Fig. 4

Figure 2 shows plots of  $Y_n(\alpha)$  for the first six spectrum numbers at  $G = 10$  (curves 1-6). Just as for other  $G$ , the stream is stable for a given Reynolds number. The perturbations with  $n = 1, 2, 4, 5$ , and  $6$  are the lower wall perturbations. The behavior of the corresponding functions  $Y_n(\alpha)$  recalls the case of a liquid at rest. It must be noted that the phase velocity of these perturbations is practically equal to zero already at  $\alpha = 0.1$ , and does not change with further increase of  $\alpha$ . All this is connected with the appearance of a flat section in the velocity profile at large  $G$  [see (1.1)]. The perturbation with  $n = 3$  will be the upper wall perturbation. The function  $Y_3(\alpha)$  has a characteristic maximum in the region of dangerous  $\alpha$ . The behavior of this function illustrates well the strong influence of the hydrodynamics of the flow on the damping decrement of the upper wall perturbations.

Just as in the case of Hartmann flow, the influence of the term  $G^2 \varphi''$  in the right-hand part of (1.2) is significant only at small  $\alpha R|C|$ ,  $\alpha R|c - 1|$ . The presence of a magnetic field leads in this case to a more rapid damping of the perturbations in accordance with the general concepts concerning the direct action of the magnetic field on the perturbations in a conducting liquid.

The considered small-perturbation spectra clearly demonstrate certain general characteristics of hydrodynamic stability of a flow with an asymmetrical profile, in distinction from a flow with a symmetrical profile [8, 9] and with an antisymmetrical velocity profile [10, 11]. We note also that this asymmetrical case differs significantly from those asymmetrical cases in which the velocity profile has a local maximum inside the channel.

4. Let us consider the behavior of small perturbations as functions of the Reynolds number. We note first that the region where the eigenvalues depend only on one parameter,  $\alpha R$  is broader than the region of applicability of the asymptotic expressions at small  $\alpha$ . This circumstance is evidenced, in particular, by the experience with numerical calculations. Thus, numerical calculations obtained for one value of  $R$  make it possible to draw a priori conclusions concerning the  $c_n(\alpha)$  dependence at other values of  $R$  in a certain region  $\alpha^2 \ll 1$ . This interval of variation of  $\alpha$  is of interest, for an appreciable realignment of the spectrum occurs in it in comparison with the asymptotic behavior. Finally, this remark can be of use in numerical

calculations for different  $R$ . (The given approximation ceases to be valid if the quantity  $c$ , after subtracting the plate velocity, becomes comparable in absolute magnitude with  $\alpha^2$ . The latter occurs, for example, on the asymptotic neutral Lin curve for a Poiseuille flow [12].)

When  $G < 6.5$ , numerical calculations show that the flow in question is stable against infinitesimally small perturbations at all Reynolds numbers, just as in the case of Couette flow in general hydrodynamics. The characteristic maximum of  $Y_1(\alpha)$  in the region of dangerous  $\alpha$  appears at  $G > 3$ , but so long as  $G < 6.5$  it does not enter into the region of instability of the  $Y$  plane. When verifying its position, as a rule, there is no need to perform numerical calculations up to very large Reynolds numbers ( $R > 10^6$ ), for with increasing  $R$  the maximum shifts toward small  $\alpha$ , where the approximation considered above can be used, naturally with due caution.

Instability sets in first at  $G = 6.5$  (the critical number  $R_*$  is in this case equal to infinity), and then, with increasing  $G$ , the value of  $R_*$  decreases monotonically to a minimum value  $5.8 \cdot 10^5$  at  $G = 10$ . Further increase of  $G$  exerts a stabilizing influence. A plot of  $R_*(G)$  is shown in Fig. 3. At  $G > 14$  one can obviously see the asymptotic behavior  $R_* = 50,000 G$ . This relation coincides with Lock's asymptotic relation [2]. Indeed, at large  $G$  the velocity profile degenerates into an exponential one, and the form of the homogeneous boundary conditions on the axis ceases to be significant.

Figure 3 shows also a plot of  $\alpha_*(G)$  ( $\alpha_*$  is the critical wave number). At  $G$  close to 6.5, the value of  $\alpha_*$  is small and vanishes at this critical value of the Hartmann number. In the case of large  $G$ , Lock's formula  $\alpha_* = 0.16 G$  holds true.

Instability is always caused only by the upper wall perturbations. The critical point lies in this case in the region of the upper plate. With increasing  $G$ , the spectral number  $n_*$  of the unstable perturbations increases. At  $G$  close to the critical value we have  $n_* = 1$ . When  $G = 10$ ,  $n_* = 3$ , as is seen from Fig. 2, which shows the successive positions of the maximum of  $Y_3(\alpha)$  at  $R = 5 \cdot 10^5$  and  $R = 10^6$  (curves 7 and 8). At  $G = 17$  we have  $n_* = 6$ .

Figure 4 shows the neutral curve for  $G = 10$ . Attention is called here to the fact that along the lower branch of the neutral oscillations the critical point does not tend to the wall when  $R \rightarrow \infty$ . The asymptotic value of  $c$  on the lower branch is a constant. In this case the eigenvalue depends only on  $\alpha R$ .

The instability in question recalls the instability of the Hartmann flow against antisymmetrical perturbations. First, this is connected with the fact that, at those  $G$  at which there is instability, the investigated velocity profile differs insignificantly from the Hartmann profile (by virtue of the relatively large value of  $G$ ). Further, if we consider the characteristic determinants [12] in both cases, then we can verify that they are approximated with a certain accuracy by one and the same expression, which does not depend on the second boundary condition: on the wall in the case under consideration ( $\varphi' = 0$ ) and on the axis ( $\varphi'' = 0$ ) in the Hartmann flow; it is difficult to estimate beforehand the ensuing error, all the more because to approximate the characteristic determinant in this case it is necessary to consider nonviscous solutions [12] in explicit form. However, if we bear in mind the numerical results obtained for the values of the critical parameters, this can be done without particular difficulty in the case  $\alpha_*^2 \ll 1$ , when the series that represent the nonviscous solutions converge rapidly. It has been established in this case that to obtain the necessary representation of the characteristic determinant it is necessary to neglect in it, in particular, the terms  $O(\varepsilon)$ ,  $\varepsilon = [(1 - c)/c]^{5/2}$ . Numerical calculations show that  $c \approx 0.85$ .

The author thanks M. A. Gol'dshtik for interest in the work and V. A. Sapozhnikov and V. N. Shtern for useful discussions.

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